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FEATURE ARTICLE

Mathematical Devices for Getting a Fair Share

Whether the problem involves an estate, a cake or an opportunity for regency, solutions now exist for obtaining an equitable division

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Since the dawn of history (and certainly before) people have bickered, even battled, over the fair distribution of resources—for example, the division of estates, disputed territories and the spoils of war. Such dilemmas have for the most part been resolved unilaterally, by kings (Solomon and the baby), judges or simple force. Few truly logical solutions evolved. In the past 50 years, however, a number of mathematical devices have been discovered that offer elegant, practical and often surprisingly simple resolutions to many problems of fair-division.

The oldest known written fair-division problem, an estate division, comes from the Babylonian Talmud of the 2nd century a.d. *(see Figure 1)*. A man dies owing 100, 200 and 300 zuz to each of three claimants, A, B and C, respectively. In most modern bankruptcy proceedings the claimants receive shares of the estate proportional to their individual claims, no matter the size of the estate. In the above example, A would always get one-sixth, B one-third and C one-half. The solution presented in the Talmud is also proportional, as long as the total estate value is 300 zuz. But if the estate is only 100 zuz, the claimants receive equal shares. Even more curiously, if the estate is 200 zuz, *A* receives 50 zuz, and *B* and *C* receive equal amounts of 75 zuz, despite the fact that their claims are not equal.

The logic of the talmudic solution remained mysterious until 1984, when the Israeli mathematicians Robert Aumann and Michael Maschler discovered that these seemingly inconsistent settlement methods actually anticipated the modern "nucleolus" solution of a three-person cooperative game. Roughly speaking, the nucleolus is the solution that minimizes the largest dissatisfaction among all possible coalitions. For example, if the total estate is 100, the talmudic solution rejects the modern proportional solution in favor of the equal-claim solution for the following reason. Any coalition of the players A, B and C gets nothing for free, since its opponent's claim totals at least the size of the whole estate. Thus with a proportionate solution, claimant A will receive 100/6, B 100/3 and C 100/2, and the maximum dissatisfaction is with claimant A, who receives less (in comparison to possible coalition claims) than he would with the equal-share solution.

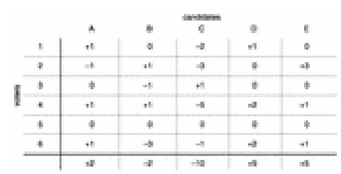
If the total estate is 200, on the other hand, a coalition of B and C against A can expect 100 "for free," since their opponent

A claims only 100 of the estate. Thus a B−C coalition can expect to share this excess 100 in addition to whatever it can gain as a team against A competing for the first 100. Because of this excess, the equal-shares solution does not minimize dissatisfaction. In this type of problem, where all objects (zuz or dollars) are valued equally by the players, many other reasonable game-theoretic solutions also exist, and no particular one seems especially compelling. In the following paragraphs, I shall introduce more fair-division problems and some solutions.

Election by Bid

Another ancient historical problem is that of selecting a new king from several candidates in such a way that each receives a fair share at the chance to become regent. Herodotus recalls the legend that Darius became king when his horse was the first to neigh as the candidates inspected city walls. An Irish legend describes how O'Neill became king under an agreed-upon rule that "he who first touches Irish soil will be monarch" as the candidates' ship approached shore: O'Neill drew his sword, chopped off his left hand and threw it ashore ahead of the others. (The Red Hand of O'Neill is still the prominent part of the O'Neill coat of arms.) Perhaps we can do better.

Modern committee and departmental structures also often call for selection of a leader, and although many voting systems have been proposed to replace the horse and hand methods, none would guarantee that every single voter is satisfied with the outcome. In the 1970s, however, Lester Dubins at the University of California, Berkeley discovered an elegant and practical selection method which does exactly that. In his solution for the problem of selecting a director or chairperson, each voter simultaneously submits a sealed bid assigning to each candidate a number reflecting the change in salary the voter agrees to accept if that candidate is selected. To preclude voters from assigning themselves large salary increases no matter who is selected, these numbers must balance and sum to zero for each voter.



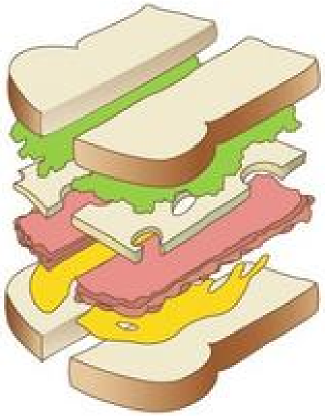
In Figure 2, for example, voter 1 does not particularly like candidate A and therefore wants an extra $1,000 to work under her. Voter 1 is indifferent to candidate B, will take a $2,000 pay cut if C is chosen director (because her friend C will give her a better office or a trip to Paris) and so forth. Voter 5 is indifferent to all candidates. The rules of the game are that these salary differentials are binding, and since each voter voluntarily sets his or her own bid, none can later claim to have been shortchanged. This is where Dubins's insight comes in. Since each row in the bid matrix sums to 0, the whole matrix sums to 0. Find the column whose sum is most negative, make that candidate director, and collect or disburse salary commitments as indicated. In this example, C would be made director, voter 1 would pay $2,000, voter 3 would receive $1,000 and so forth. Not only does each voter receive the salary he or she suggested, but there is even a surplus amounting to that column sum (here $10,000) to be distributed among the voters. Each one receives a salary strictly higher than he or she agreed to accept!

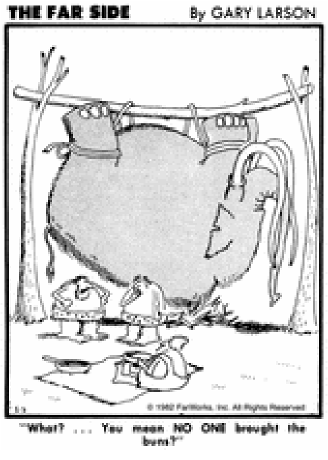
An even older and more basic fair-division problem is the prehistoric question of how to divide an object such as a cake or a mastodon among several people *(Figure 3).* If there are only two claimants, and the object is bilaterally symmetrical, then a sharp scalpel will allow a fair division. To divide an inhomogeneous, irregular-shaped object between two people, the time-honored "one cuts, the other chooses" solution guarantees each person a portion he feels is fair, even though the participants may have different values.

Two crucial assumptions are necessary to guarantee the success of the cut-and-choose method. First, the object must be *continuously divisible* by the cutter—or at the very least, he must be able to divide it into what he considers to be equal shares. And second, the values of each player must be *additive*: If a player values a certain piece at 40 percent, he must value the remainder at 60 percent of the total value. In many political or social problems, this crucial additivity condition does not hold, so the cut-and-choose algorithm will not succeed. The value of all of Jerusalem to the Israelis or the Palestinians is much greater than the sum of its parts, and Solomon capitalized on a nonadditivity hypothesis in his famous baby-division problem.

The Ham Sandwich Theorem

The problem of fairly dividing a single object or a group of objects among more than two people (for example, victuals among three cave dwellers, as in Figure 3), even when the object is completely divisible and the values additive, remained unresolved until the 1940s, when the Polish mathematician Hugo Steinhaus made two important discoveries that have inspired much of modern research on fair division. First, he proved in his famous Ham Sandwich Theorem that any three three-dimensional objects (say ham, cheese and bread) may be simultaneously bisected by a single plane *(see Figure 4).* The objects need not be connected, regular in shape or in any special orientation; more generally, any *n* objects in *n*-dimensional space may be simultaneously bisected by a single hyperplane.



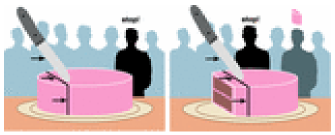


"What? . . . You mean NO ONE brought the buns?"

In two dimensions, for example, this theorem says that if salt and pepper are sprinkled randomly on a table, a single straight line always exists that will simultaneously separate the salt into two equal parts and the pepper into two equal parts. On the other hand, there is not always a line that will simultaneously bisect salt, pepper and sugar on a tabletop (for example, when the salt is all sprinkled tightly around one vertex of a triangle, the pepper around a second vertex and the sugar around the third). It is crucial that the number of objects not exceed the spatial dimension. So, although Steinhaus's theorem says that a Big Mac may be sliced with a straight (planar) sweep of a knife so that it simultaneously bisects the bread, meat and cheese, there is no guarantee that any planar bisection also contains equal amounts of the lettuce or "secret sauce." (The example of the Big Mac was chosen to emphasize that even if the bread, meat and cheese are in several pieces, there is still a planar cut that simultaneously bisects these three, or any three, ingredients.)

The Ham Sandwich Theorem suffers one serious additional drawback beyond the dimensional restriction: It is not constructive. That is, although it guarantees that a sandwich bisection exists, it gives no clue as to how to find one. This distinction between nonconstructive and constructive proofs is an important one in mathematics, especially in applications. A nonconstructive existence proof argues indirectly, as in "if the temperature at noon yesterday was 20 degrees and it is now 30 degrees, then sometime in between it must have been exactly 25 degrees." The argument gives no indication of when it was 25 degrees.

A constructive proof, on the other hand, argues existence by providing an algorithm or procedure for finding the object in question, and this is exactly what Steinhaus's second major contribution did. It proves the existence of fair divisions by giving a practical and general procedure for dividing an inhomogeneous irregular object such as fruitcake among an arbitrary number of people so that each receives a portion he considers a fair share. Each of *n* people will receive a piece he values at least one-*n*th of the cake, even though different individuals may have different values—one preferring the frosting, another the nuts and so forth. Since Steinhaus's discovery, a number of similar algorithms have been devised based on ideas of rotating reduction, iterated cut-and-choose and elegant sliding-knife techniques *(see Figure 5).*



Sliding-knife schemes have been criticized as impractical because they require continuous evaluation of the pieces as the knife moves and in this sense do not end after a number of finite steps. An alternative cake-division solution that does end after a finite number of steps (a criterion often viewed as essential to computer implementation) is called the "round-table" solution. Suppose *n* Girl Scouts sit at a round table, with the cake in the center. One (any) scout begins by cutting a piece she feels is a fair share and then passes it to the person on the left. If that person feels that this piece is less than or exactly equal to a fair share, she passes it to the person on her left, but if she feels that it is more than one *n*th, she cuts off a piece so that she feels the remainder is exactly one *n*th, returns the cutoff excess to the center of the table and passes the reduced piece to her left. This continues until everyone has passed without cutting, and the last person to cut the piece receives it. Thus the first piece is given away after at most *n* steps, and the procedure then continues for the other $n-1$ people.

Notice that both the sliding-knife and round-table solutions assume that none of the participants is bluffing about his values. If bluffing is allowed, an element of risk is introduced to the problem. A player who bluffs by letting the knife continue past what he considers a fair share, in hopes of claiming a larger-than-fair share, is also risking receiving a smaller-than-fair share if another player says "stop" too soon. All the risk-free fair-division methods being discussed here, which guarantee certain sizes of shares, also have game-theoretic versions that incorporate various elements of risk resulting from random play and bluffing.

Fair Division for Disarmament

In the 1980s, mathematician Robion Kirby at Berkeley proposed an elegant and practical application of these fair-division algorithms to the problem of disarmament. Suppose countries A and B agree to a 50-percent arms reduction; Kirby's method of accomplishing that works as follows. Country A openly declares the relative values of each of its arms, and country B selects that 50 percent of A's declared values that it wants destroyed. Simultaneously, B declares the values of its own arms, and A picks the 50 percent it wants destroyed. This method guarantees that each country will be satisfied that it has destroyed *more* than half of the other's armaments (except in the rare case that both countries value every weapon exactly the same). And if both countries declare the true values of their armaments, each is also guaranteed that it has lost only half. If either country lies or bluffs, it risks losing more.

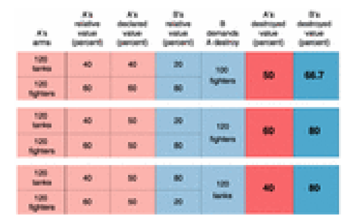
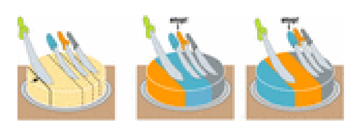


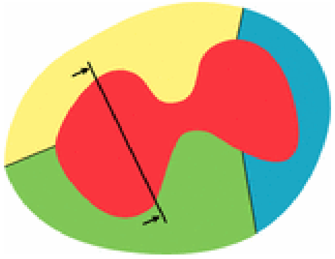
Figure 6 shows a typical example of how the destruction of A's weapons proceeds. A's weapons consist of 120 tanks and 120 fighters, of which A values the fighters half again as much as it does the tanks. If B thinks the tanks are worth more than A does—say B thinks the tanks are 45 percent of A's weaponry and the fighters are 55 percent (bottom case in Figure 6)—then B will choose to have as many tanks destroyed as possible, the remainder to come from fighters. In this case, B will demand that A destroy all its own tanks (for 40 percent of A's declared value), plus 20 fighters for the remaining 10 percent. A thinks it has lost 50 percent of its weaponry, but B thinks that A as has destroyed 120/120 x 45 percent + 20/120 x 55 percent =~ 54.17 percent. Kirby's method not only guarantees each side what it considers a fair reduction, but it also does so without the need for long negotiations over the values of each type of weapon.

Divvying Up Territory

The fair division of land introduces topological complications not inherent in cake division, because, unlike pieces of cake, land may not be repositioned arbitrarily and because location is an issue. With division of land, a typical requirement is that each participant receives a portion that is adjacent to its own homeland, rather than an inaccessible island in the midst of enemy territory. Here the sliding-knife solution may fail *(see Figure 8)*, as will other standard fair-division algorithms. Using a convexity theorem for measures, I proved there is always a land-division solution in which each country receives a fair share consisting of a single piece of land adjacent to its own territory. That solution, however, was nonconstructive, but several years later Anatole Beck of the University of Wisconsin, Madison discovered an ingenious and complicated constructive algorithm for dividing land fairly.



The convexity theorem for measures, proved by Russian mathematician A. Lyapounov in 1940, lays the groundwork for many land-division solutions. It says, for example, that if the proportions of the various ingredients (salt, sugar, fat, flour, etc.) of a cake are plotted for every conceivable piece of the cake, then the resulting region will always be convex—that is, it will be a shape without dents or holes. The power of this celebrated theorem has been applied to solve many famous problems in mathematics, among them the bang-bang principle of



optimal control theory and another fair-division problem, R. A. Fisher's 1930s "problem of the Nile."

The problem of the Nile concerns the fair division of land along the banks of a river that is subject to periodic flooding. Since the value of any given tract of land each year depends heavily on the most recent flood height (some heights bring a new layer of topsoil; others deplete it), the question was whether there is always a way to give each family a fixed plot of land so that every plot gains or loses exactly the same value no matter the height of the flood. That is, can each family be given deed to a single piece of land so that if the flood height one year results in a decrease of 10 percent in value of one family's plot, then every family's plot decreases 10 percent in value in that year; likewise, if the flood height was such that one family's plot increased in value 20 percent, then every family's plot increases 20 percent in value? In cake-cutting terms, this asks whether a cake may be cut in such a way that each piece contains exactly the same amount of calories, fat, sugar and so on. Jerzy Neyman showed in 1949 that Lyapounov's theorem implies that such divisions do always exist, although again the solution is nonconstructive, and no practical solution analogous to the sliding-knife method has yet been found.

A step toward that solution was taken in 1980 by Walter Stromquist, a mathematician with Wagner Associates, who found an intriguing variation on the sliding-knife method that gives each of three people a piece he prefers to all other pieces. Neyman's solution to the problem of the Nile, although nonconstructive, guarantees that such partitions exist and, in fact, that there are ways to divide the cake so that every person values each piece exactly the same. The classical sliding-knife solution, on the other hand, usually does not yield such an "envy-free" division, even for three people. For example, if player 1 values only raisins, and the cake contains exactly three raisins, he will stop as soon as the sliding knife passes the first raisin. The subsequent division of the remainder of the cake by the other two (perhaps raisin-indifferent) players may well be one in which one of their pieces contains two raisins. So, although player 1 will receive a piece he considers a fair share (at least one-third), he would prefer to have the piece with two raisins.

More-than-Fair Divisions

Since 1980, a number of other envy-free algorithms have been found, including a constructive but complex envy-free division scheme by New York University political scientist Steven Brams and Union College mathematician Alan Taylor. Envy-free solutions, however, guarantee only a fair share, whereas in fact *super-fair* divisions usually exist. If three people are to divide a three-layer cake of which the first person values only the first layer, the second the second layer and the third the third, then any envy-free solution such as Stromquist's, which makes vertical slices, will necessarily give each person a piece he values exactly one-third. A horizontal-slice partition, however, would give each player his desired layer, leaving each feeling he has received everything of value.

Super-fair divisions, in which each player receives a share he feels is worth strictly *more* than a fair share, are in fact possible in every problem in which there is at least one piece of cake not valued equally by everyone. This is especially easy to believe in the case of two people, since their different values imply that there is some piece that one values more highly than the other does. Giving each person the piece he prefers is a start toward a super-fair partition. For three or more people the existence of super-fair divisions is not at all obvious, but it does follow as another consequence of Lyapounov convexity. Polish mathematician K. Urbanik, and independently Dubins and Edward Spanier at Berkeley, proved that if any two of the participants' values differ on even the tiniest of pieces, then there is always a super-fair partition.

John Elton, Bob Kertz (both at Georgia Institute of Technology) and I were able to quantify exactly how much more than a fair share is possible as a function of the *cooperative value, M,* of the cake. If n people are to divide a cake and every piece is given to the person who values it most, M is the sum of each person's resulting perceived share. (Informally, if each player pays into a common account his value of the piece he receives, M is the balance in the account.) We proved there is always a solution in which each person receives at least $1/(n - M + 1)$ of the total value. Since M is strictly larger than

1 (except when all values for all players are identical), this new guarantee is strictly larger than the fair share 1/*n.* For example, if three people are to share a cake whose cooperative value *M* they place at 3/2, then a partition is possible guaranteeing each of the three people a piece he values at least 1/(3 − 3/2 + 1) = 40 percent of the total cake. In general this is the best that can be expected under those conditions.

Dividing the Indivisible

All the fair divisions described above depend heavily on the complete divisibility of the object to be partitioned, whereas in real life the object may consist of indivisible pieces. Even a real cake has basic indivisible components (crumbs, perhaps, or at least molecules), so officially speaking even the sliding knife does not perform perfectly. More seriously, many estate settlements consist entirely of indivisible objects such as pianos or pieces of silverware, and sliding-knife solutions clearly will not work.

When fair-divisions problems have value measures with indivisible "atoms," I have discovered the exact minimally guaranteed portions as a function of atom size. For example, if three people are to divide a cake, and each agrees that no crumb is worth more than one-thousandth of the whole cake, then there is always a partitioning so that each person receives a piece he values at least 83/250, which is just slightly less than the guaranteed 1/3 share possible if the sliding knife could also split crumbs and molecules. However, since the last crumb may not be further divided but must instead be given in its entirety to one of the players, this means some player may receive strictly less than a fair share. The significance of the number 83/250 is that this is the new universally guaranteed share, in place of 1/3, in every division problem involving three people and atoms of size one-thousandth. Optimal share values have been found for all *n* and all atom sizes, and although the function describing them is somewhat complicated, with unexpected sharp points and a fractal-like (self-similar) shape, it is explicit and easy to evaluate.

Similarly, Elton and the I recently found a generalization of Lyapounov's basic 1940 convexity theorem. This gives the maximum "hole size" in the range as a function of the maximum atom size. Measures or values with indivisible portions do not satisfy the hypotheses of Lyapounov's theorem and hence may have ranges with dents or holes, but the size of the largest possible hole or dent can now be determined explicitly by atom size. This allows the extension of the general bang-bang principle—in many control problems, such as moving an object in space from point A to point B in minimum time, the optimal control is a series of full-throttle and full-brake maneuvers—to systems with discontinuities and offers new insights into super-fair divisions.

Super-Fair Lotteries

Distributing indivisible objects from an estate to people whose relative and total values of the objects may differ is perhaps the most difficult fair-division problem. Classical solutions usually involve selling the objects and distributing the money, employment of an outside judge or assessor, or use of side payments (from claimant to claimant). The first solution, however, ignores the individual values of the participants, as does use of a judge, who introduces a new value judgment of his own. Side-payments are only feasible if each participant has sufficient resources to make competitive bids.

One alternative that avoids these shortcomings is the use of lotteries, a technique that introduces elements of randomness and risk. Suppose a man wills two indivisible objects, such as a Stradivarius violin and a thoroughbred racehorse, equally to his daughter and son. Both agree that the violin is worth more than the horse, but one places its value at 70 percent of the total estate, whereas the second values it at only 60 percent. One possible lottery would be a toss of a fair coin with the winner receiving the violin and the loser the horse, and another would be separate tosses of the coin for each object. These lotteries are both fair in the sense that the e*xpected value* each person receives is exactly 50 percent of the estate.

(The expected value of a random variable is the long-run average gain if the lottery is performed many times.) In this case, a large number of repetitions would result in each player getting the violin about half the time and the horse half the time, so the long-run average gain of each player is half the value of the violin plus half the value of the horse, which by additivity is exactly 50 percent, regardless of their values of each.

In fact, it is possible to do better. Steve Demko at Iterated Systems and I have shown that super-fair lotteries exist for all such problems, and we have identified the optimal lotteries. In the violin-horse example, an optimal lottery is to give the horse to player 2 and then have a lottery in which player 1 wins the violin with a probability of 10/13, player 2 winning otherwise. This super-fair lottery gives an expected value of 7/13 to each player: Player 1 receives the violin, which she values at 0.7, 10/13 of the time, for an expected value of $(0.7) \times 10/13 = 7/13$; player 2 always receives the horse, which he values at 0.4, and also receives the violin 3/13 of the time, for a long-run average of $0.4 + (0.6) \times 3/13 = 7/13$. Moreover, this particular lottery is optimal in that no other lottery gives a higher expected value to both players. Such a solution avoids the use of side payments or outside judges while respecting the different values of the players, but it does introduce an element of risk not present in the fair-division devices presented earlier—something that may be discomforting to certain people.

How can estates be divided in a practical and fair way? How can a cake or territory be divided so that each participant receives the best-possible solution? Although there has been progress, no good solutions have yet been found for many of these ancient problems, and the ideas continue to challenge and inspire mathematicians.